Classification of compact analytic manifolds over non-Archimedean locally compact fields

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January 15, 2022

Abstract

We complete a proof sketched in [Serre, 1992] of a theorem which completely classifies the topological structure of compact analytic manifolds over a locally compact non-Archimedean field.

Contents

1 Introduction

A field k equipped with an absolute value $|\cdot| : k \to \mathbb{R}_{\geq 0}$ is called non-Archimedean if it satisfies the stronger triangle inequality: $|a + b| \leq \max\{|a|, |b|\}$. The most prominent examples of such fields are the p-adics $\mathbb{Q}_p$. The stronger triangle inequality has a drastic impact on the geometry of such fields - for example, any two open balls in k are either disjoint or one is contained in the other. In fact, when k is locally compact (as in the case of the p-adics), any compact analytic manifold defined over k decomposes into a finite disjoint union of balls, the number of which is uniquely determined modulo a constant depending on k. This result is shown in theorem 3. Before arriving at this proof, we present a reminder of the relevant parts of the theory of non-Archimedean fields and analytic functions over such fields. The focus here is on building a clearer mental picture rather than giving proofs, which are often routine and available in a good reference on the subject.

Some basic knowledge of manifolds and commutative algebra will be assumed throughout, and familiarity with non-Archimedean fields will be helpful. Most theorems proven here were first shown in [Serre, 1992], but the exposition has been expanded upon in many proofs compared to the source in the hopes of added clarity.

Acknowledgements

The author would like to thank Professor Johannes Nicaise for his guidance in this project, and for suggesting the use of lemma 5 in proving theorem 3.

2 Properties of k

Firstly, we fix some notation. Recall that a non-Archimedean absolute value on a field k is a function $|\cdot| : k \to \mathbb{R}_{\geq 0}$ such that for all $x, y \in k$:

1. $|x| = 0 \iff x = 0$
2. $|xy| = |x| \cdot |y|$
3. $|x + y| \leq \max\{|x|, |y|\}$

Now let k be a locally compact field with a non-trivial, non-Archimedean absolute value, where k is implicitly endowed with the metric topology induced by setting $d(x, y) = |x - y|$. Such a field is also called a *local field*. A closed ball of radius r centered at a is denoted $\overline{B}(a, r)$, while the open ball is denoted $B(a, r)$. The notion of balls is extended to k^n by defining a closed ball in k^n of radius r centered at a to be the set:

$$\{x \in k^n \mid |x_i - a_i| \leq r,\ 1 \leq i \leq n\}$$

Open balls in k^n are defined similarly. Denote the closed unit ball in k^n by $\overline{B}(0, 1)^n$.

We will now state without proof some elementary properties of non-Archimedean local fields, with the aim of building intuition. A reference for these facts is [Gouvea, 2003], albeit considered in the case of the p-adics.

Recall that a valuation on a field k is a map $v : k \to \mathbb{R} \cup \{\infty\}$ which defines a group homomorphism $k^\times \to (\mathbb{R}, +)$ and maps $v(0) = \infty$. For a non-Archimedean field k, the absolute value defines a valuation by setting $v(x) = -\log|x|$. If k is locally compact and non-Archimedean, the following statements hold:

1. k is complete with respect to its absolute value.

2. $\nu : k \to \mathbb{R} \cup \{\infty\}$ given by $\nu(x) = -\log|x|$ is a valuation on k with image a discrete subgroup of $(\mathbb{R}, +)$ (i.e. isomorphic to $\mathbb{Z}$).

3. $\mathcal{O}_k = \{x \in k \mid \nu(x) \geq 0\} = \overline{B}(0, 1)$ is a ring called the *valuation ring* of k. It is a discrete valuation ring with a unique maximal ideal $\mathfrak{m} = B(0, 1)$.

4. $\mathfrak{m}$ is generated as an ideal by an element π called the *uniformizer*. Every element $a \in k$ has the property that $|a| = |\pi^m|$ for some $m \in \mathbb{Z}$.

The final property corresponds to the fact that by choosing a suitable base for the logarithm when defining ν, we have that $\nu(\pi) = 1$. Note that the image of any integer $n \in \mathbb{Z}$ in k lies in $\mathcal{O}_k$. We will freely switch between the notation $\overline{B}(0, 1)$ and $\mathcal{O}_k$ for the closed unit ball and $B(0, 1)$ and $\mathfrak{m}$ for the open unit ball, in order to emphasise the topological or algebraic properties respectively.

A further property of such fields is that the *residue field* $\tilde{k} = \mathcal{O}_k/\mathfrak{m}$ is a finite field; to show this, we explore some properties of open and closed balls in k.

2.1 Properties of balls in k

Lemma 1. *[Gouvea, 2003, Proposition 2.3.6] Open and closed balls in k have the following properties:*

1. *If $b \in B(a, r)$, then $B(b, r) = B(a, r)$ (every point in an open ball can be taken as its center).*

2. *All open balls are closed.*

3. *All closed balls of radius $r > 0$ are open.*

4. *Any two open (resp. closed) balls are either disjoint or one is contained in the other.*

Proof. Omitted; see *loc. cit.* □

All of these properties rely only on the non-Archimedean nature of the absolute value. Although it should not be taken as a proof, properties (2) and (3) may be easier to think about when we have local compactness, since then the value group is discrete so any open ball $B(0, |\pi^m|)$ is equal to the closed ball $\overline{B}(0, |\pi^{m+1}|)$. In any case, in the sequel, we will use 'ball' to simply refer to any open or closed ball of non-zero radius. We also note that since a ball in k^n is a finite product of balls in k, these properties all extend to the $n \geq 1$ dimensional case.

Since k is locally compact, it follows that 0 is contained in a compact ball. Then, any ball is the image of a translation and scaling of this ball, hence is compact since these operations are homemorphisms.

In particular then, the closed unit ball $\overline{B}(0, 1)$ is compact. For each $x \in \overline{B}(0, 1)$, by property (1) in lemma 1, we have that $B(x, 1) \subset \overline{B}(0, 1)$. The open balls $\{B(x, 1) \mid x \in \overline{B}(0, 1)\}$ form an open cover of $\overline{B}(0, 1)$, hence have a finite subcover. We may choose a set of centers $\{a_1, a_2, \ldots, a_q\}$ for this subcover. Then, each element of $\tilde{k} = \mathcal{O}_k/\mathfrak{m}$ is represented by an element a_i for some $1 \leq i \leq q$. Hence $\tilde{k}$ is a finite field; for the rest of this document, q refers to the size of $\tilde{k}$.

Our final lemma of the subsection was stated in a weaker form in [Serre, 1992, pp 98], but the proof is essentially the same.

Lemma 2. *Serre [1992] Let U be a closed and open set of a ball $B \subseteq k^n$, where B has radius r'. Then there exists some $0 < r < r'$ such that for any $0 < s \leq r$, U is the disjoint union of a finite number of balls of radius s.*

Proof. Let $V = B - U$ so that $\{U, V\}$ is an open cover of B. As B is a compact metric space with respect to the metric $d(x, y) = \max_i\{|x_i - y_i|\}$, by Lebesgue's number lemma, there exists a radius $r > 0$ so that every ball of radius less than or equal to r is contained in either U or V. Therefore, for any $0 < s \leq r$, we may cover U by a union of balls of radius s. Since U is a closed subset of B it is also compact, so this cover can be taken to be finite and necessarily disjoint. □

2.2 Visualising k

The properties shown so far can often have unintuitive consequences: for example, the unit sphere $\overline{B}(0, 1) - B(0, 1)$ is an open and closed set. It is also *not* the boundary of the open unit ball, which is boundaryless as its closure and interior are both itself. It is therefore useful to attempt to provide some visualisation of the fields, as the usual mental model of an Archimedean field such as $\mathbb{R}$ or $\mathbb{C}$ can be misleading for us. To do this, we will consider the example of the p-adic field $\mathbb{Q}_p$. As a set, an element $x \in \mathbb{Q}_p^\times$ can be written uniquely as a formal summation:

$$x = \sum_{i=n}^{\infty} a_i p^i$$

where $n \in \mathbb{Z}$ and for each i, a_i is an integer in the range $0 \leq a_i \leq p - 1$. The number n is in fact the p-adic valuation of x, i.e. $\nu_p(x) = n$, which extends to 0 by setting $\nu(0) = \infty$. The absolute value on $\mathbb{Q}_p$ is defined as $|x| = p^{-\nu_p(x)}$, from which it is clear that the uniformizer can be taken to be $\pi = p$.

We will further restrict our attempts to the valuation ring $\mathbb{Z}_3$ of $\mathbb{Q}_3$. From the preceding description, we can represent an element of $\mathbb{Z}_3$ as an infinite sequence of integers $(a_0, a_1, \ldots)$. For each i, we have that $a_i \in \{0, 1, 2\}$. It follows that $\mathbb{Z}_3$ can be seen as a tree of infinite depth, where each edge corresponds to an element of $\{0, 1, 2\}$ and a node $\eta_{a_0 \ldots a_i}$ of depth $i \geq 1$ is given by following the path $a_0 \ldots a_i$. The leaves of the tree then represent elements of $\mathbb{Z}_p$. This is shown in figure 1.

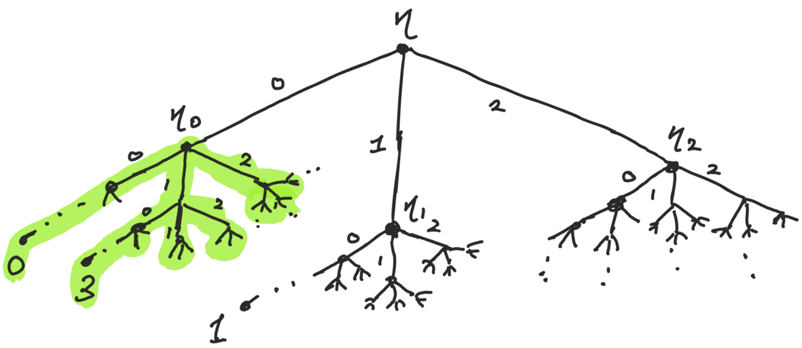

Figure 1: $\mathbb{Z}_3 = \overline{B}(0, 1) \subseteq \mathbb{Q}_3$ with the open ball $B(0, 1)$ highlighted. The complement is the unit sphere.

This image is not accurate with regards to the topology of the field: indeed, $\mathbb{Q}_3$ is totally disconnected, whereas the tree shown is path connected. Nonetheless, this captures some of the important properties shown so far. For example, figure 1 shows how the closed unit ball decomposes as $\overline{B}(0, 1) = B(0, 1) \sqcup B(1, 1) \sqcup B(2, 1)$. Any ball in $\mathbb{Z}_3$ can be considered as a subtree rooted at some node $\eta_{a_0 \ldots a_i}$, which allows us to visualise property (4) in lemma 1. The fact that any point in a ball can be taken as its center is represented by a symmetry of the tree, seen in figure 2. Figure 3 indicates how $|x - y|$ is computed. There is a unique path starting at x, ascending up the tree to some node η_{xy} before descending down the tree to the node y. Then, $\nu_p(x - y)$ is precisely

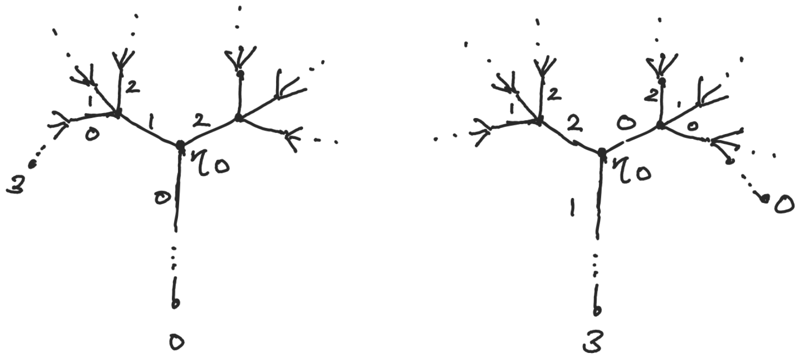

Figure 2: By thinking of the center of the ball as the root of the corresponding tree, we see that $B(0, 1)$ can be 'centered' at both 0 and 3.

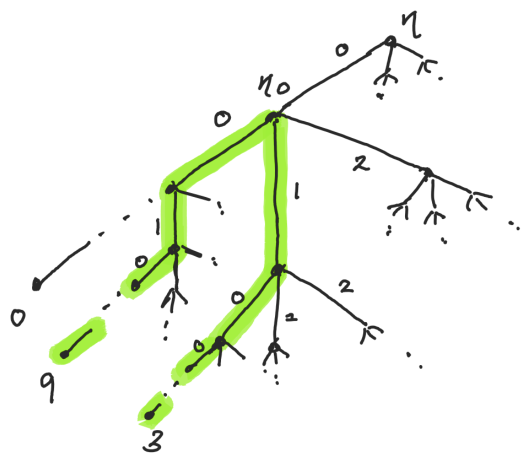

Figure 3: The value $v_p(x - y)$ can be computed by considering the unique path between x and y.

the depth of the node η_{xy}. In the case shown by the figure, $\eta_{xy} = \eta_0$ has depth 1, which indicates that $|9-3| = 3^{-1}$. Indeed, $9-3=6$ corresponds to the sequence $(0,2,0,\dots)$.

3 Properties of k-analytic functions

Recall that a map $f : U \subseteq k^n \to k$ is said to be *analytic* on the open subset U if at each $x \in k$, there exists an open ball $B \subseteq U$ centered at some a containing x such that f restricted to B is given by a power series converging on B:

$$f(x) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha (X-a)^\alpha$$

where $(X-a)^\alpha = (X_1-a_1)^{\alpha}_1 \dots (X_n-a_n)^{\alpha_n}$. A function $f : k^n \to k^n$ is analytic if each component function is. We will use the notation $|\alpha|$ to denote $\alpha_1 + \cdots + \alpha_n$ for any multi-index $\alpha \in \mathbb{N}^n$.

In order to effectively study analytic functions, it is useful to have some basic results on sequences and the convergence of series. We will once more state some results relevant to us without proof.

Lemma 3. *[Gouvea, 2003] Let $(a_n)_{n\in\mathbb{N}}$ be a sequence.*

1. *$(a_n)_{n\in\mathbb{N}}$ is Cauchy (equivalently, convergent as k is complete) if and only if*

$$\lim_{n\to\infty} |a_{n+1} - a_n| = 0$$

2. *A series $\sum_{n\geq 0} a_n$ over k converges if and only if $\lim_{n\to\infty} a_n = 0$.*

3. *If the series $\sum_{n\geq 0} a_n$ over k converges to a value a, then the strong triangle inequality generalises to say that $|a| \leq \max_n |a_n|$.*

Proof. Omitted; see lemma 4.1.1 and corollary 4.1.2 in *op. cit.* □

This is an important difference between real and non-Archimedean analysis - over $\mathbb{R}$, the series $\sum_n 1/n$ is a counterexample to statement (2) above.

When working with power series, it is useful to be able to recenter them; a tool to help us in this case is captured by the following lemma, which allows us to rearrange double summations in sufficiently nice cases.

Lemma 4. *[Gouvea, 2003, Proposition 4.1.4] Suppose that a sequence $(b_{ij})_{i,j\in\mathbb{N}}$ in k has the following properties:*

1. *For every i, $\lim_{j\to\infty} b_{ij} = 0$.*

2. *For all $\epsilon > 0$ there exists an N independent of j such that $|b_{ij}| < \epsilon$ whenever $i \geq N$.*

Then we have that the series

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} b_{ij} \qquad \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} b_{ij}$$

both converge to the same value.

Proof. Omitted. This is shown in *loc. cit.* for the case where $k = \mathbb{Q}_p$ is a p-adic field, but the proof only uses properties of general non-Archimedean fields. □

It is interesting to note that when re-centering a convergent power series to a point at which it converges, the region of convergence does not change [Gouvea, 2003, Proposition 4.4.2]; we can intuitively see this as a consequence of property (4) of lemma 1.

Finally, we state an inverse function theorem for power series in many variables over k - it is in fact a corollary of an implicit function theorem for non-Archimedean fields. Denote by $\mathcal{O}_k[[X_1, \dots, X_n]]$ the ring of power series in n variables with coefficients in $\mathcal{O}_k$. $J_f(a)$ denotes the Jacobian of an analytic function $f : k^n \to k^n$ at $a \in k^n$, and $\det J_f(a)$ is its determinant. For $f = (f_1, \dots, f_n)$ where for each i we have $f_i \in \mathcal{O}_k[[X_1, \dots, X_n]]$, we say that $f_i(X) = \sum_{\alpha \in \mathbb{N}^n} c_{i,\alpha} X^\alpha$ is given by a *special restricted power series* if $f_i(0) = 0$ and $c_{i,\alpha} \equiv 0 \mod \pi^{|\alpha|-1}$. The final condition can also be stated 'topologically' as $|c_{i,\alpha}| \leq |\pi^{|\alpha|-1}|$. It is easy to see that any special restricted power series converges on $\overline{B}(0,1)^n$.

Lemma 5. *[Igusa, 2000, Corollary 2.2.1] Suppose $g = (g_1, \dots, g_n)$ is such that each $g_i(x)$ is a special restricted power series, $g(0) = 0$ and $|\det J_g(0)| = 1$. Then there exists a unique $f = (f_1, \dots, f_n)$ such that $g(f(x)) = x$, where f_i is also a special restricted power series for each $1 \leq i \leq n$ and such that f is an analytic automorphism of $\overline{B}(0,1)^n$.*

Proof. Omitted; see *loc. cit.* □

4 The classification theorem

We work towards the proof of the classification theorem by proving some lemmas pertaining to (analytic) manifolds over k, which will be useful in showing the main theorem. For our purposes, a manifold is a second countable, Hausdorff topological manifold of constant dimension, equipped with a k-analytic structure. From now on, X denotes an analytic manifold of dimension n over k. Define a subset $B \subseteq X$ to be a ball if there is some chart (U, ϕ) containing B such that $\phi(B)$ is a ball in k^n.

Lemma 6. *Let $U \subseteq X$ be a ball. Then U is the disjoint union of q^i balls for some $i \in \mathbb{Z}_{>0}$.*

Proof. We may assume that there exists an analytic isomorphism $\psi : U \to \psi(U)$ where $B = \psi(U)$ is equal to ball $\overline{B}(0,1)^n \subseteq k^n$. Since the residue field is finite of size q, $\overline{B}(0,1)$ is the disjoint union of q translates of $B(0,1)$. It then follows that B is the disjoint union of sets of the form $B(x_1, 1) \times \cdots \times B(x_n, 1)$, for some $x_i \in k$, $1 \leq i \leq n$. But such a set is precisely the open ball of radius 1 at the point $(x_1, \dots, x_n) \in k^n$, and there are q^n such open balls. The lemma follows by taking the preimages of ψ on these balls. □

It follows that by working in a chart and repeating this process, we can in fact subdivide U into balls of arbitrarily small radius.

Next, we state a result which essentially forms one half of the main theorem. Recall that an open cover of a space X is said to be *locally finite* if each point has a neighbourhood intersecting only finitely many elements of the cover. X is said to be *paracompact* if every open cover has a refinement which is a locally finite open cover.

Theorem 1. *[Serre, 1992] If X is paracompact, then it is the disjoint union of balls.*

Proof. Assume X is paracompact and let $\{U_\lambda\}_{\lambda \in L}$ be a covering by balls with locally finite open refinement $\{V_\mu\}_{\mu \in M}$. We claim without proof that there exists a locally finite closed refinement $\{W_\nu\}_{\nu \in N}$ of $\{V_\mu\}$ (see [Michael, 1953, lemma 1] for a proof). For each $\nu \in N$, there exists some

$\mu \in M$ and $\lambda \in L$ such that $W_\nu \subset V_\mu \subset U_\lambda$. It follows that W_ν is compact since it is a closed subset of the compact set U_λ. We may then cover W_ν by finitely many balls $\{B_{\nu,i}\}_{i \in I_\nu}$ where for all $i \in I_\nu$ we have $B_{\nu,i} \subset V_\mu$. Then $\{B_{\nu,i}\}_{\nu \in N, i \in I_\nu}$ is a locally finite covering by balls, since each ball meets finitely many V_μ and hence finitely many other balls in the cover.

Denote this cover by $\{U_i\}_{i \in I}$. If $F(I) = \{J \subseteq I \mid J \text{ finite}\}$, then define for $J \in F(I)$:

$$V_J = X - \bigcup_{j \notin J} U_j$$
$$U_J = \bigcap_{i \in J} U_i \cap V_J$$

We see that V_J is empty or otherwise open and compact, since it is equal to the union of $\bigcap_{j \notin J}(U_i - U_j)$ over $i \in J$. Each U_i meets only finitely many of the U_j by the earlier remark. Then, U_J is an open and compact (hence closed) subset of a ball (when non-empty), so it is a disjoint union of a finite number of balls by lemma 2. By construction, the U_J are disjoint for $J \in F(I)$, giving the desired result. □

In fact, the previous lemma is really an if and only if, since balls are compact and a disjoint union of compact sets is paracompact.

We are now ready to prove the classification theorem for compact analytic manifolds over k. In order to do so, we first consider a special case which was explicitly proven in [Serre, 1992]; here it is presented in theorem 2 below.

Theorem 2. *[Serre, 1992, pp 99-100] Let $X \subseteq k^n$ be a ball. Suppose for some $n > 0$, $\{U_i\}_{i=1}^n$ are balls in k^n and L_i are linear isomorphisms so that $L_i(U_i) \subseteq X$ for each i, and X decomposes as a disjoint union $X = \coprod_{i=1}^n L_i(U_i)$. Then $n \equiv 1 \mod (q-1)$.*

Proof. Firstly, suppose that we have $U = \overline{B}(0,1)^n$ and L is a linear isomorphism of U onto $L(U)$, such that $L \in M_n(\mathcal{O}_k)$. Then in particular, $L(U)$ is an open and closed subset of the ball $\overline{B}(0,1)^n$. Now note that by lemma 2, there exists some radius r so that for all $0 < s \le r$, $L(U)$ is a disjoint union of balls of radius s. Fix a value $0 < s \le r$ and let h be the number of balls in the decomposition of $L(U)$ into balls of radius s.

We note that for any positive integer μ, the number of cosets of $\mathfrak{m}^\mu$ in $\mathcal{O}_k$ is precisely the size of $\mathcal{O}_k/\mathfrak{m}^\mu$, which is q^μ. Now let $\mu > 0$ be such that $\mathfrak{m}^\mu$ is the ball of radius s in $\mathcal{O}_k$ around 0. Then $(\mathfrak{m}^\mu)^n$ is the open ball of radius s in $\overline{B}(0,1)^n$ around 0, so that there are a total of $q^{\mu \cdot n}$ balls of radius s in $\overline{B}(0,1)^n$. Note that in the preceding statement, $\mathfrak{m}^\mu$ is the μ-th power of $\mathfrak{m}$ as an ideal, while $(\mathfrak{m}^\mu)^n$ represents the n-th Cartesian product of the ideal, so that it is a subset of $\overline{B}(0,1)^n$.

Next, we see that we can write $\mathcal{O}_k^n := \overline{B}(0,1)^n$ as a disjoint union of g translates of $L(U)$, by considering the cosets of $L(U)$ in $\mathcal{O}_k^n$. Indeed, $g = |\mathcal{O}_k^n/L(U)|$, where the quotient is taken in the sense of $\mathcal{O}_k$-modules. Then, $\mathcal{O}_k^n/L(U)$ is a torsion module over $\mathcal{O}_k$, since multiplying any element of $\mathcal{O}_k^n$ by a suitable power of the uniformizer π annihilates it. Since $\mathcal{O}_k$ is a discrete valuation ring, it is a principal ideal domain, and all ideals are of the form $\mathfrak{m}^k$ for some k, so we may apply the structure theorem for finitely generated modules over a principal ideal domain to write:

$$\mathcal{O}_k^n/L(U) \cong \mathcal{O}_k/\mathfrak{m}^{k_1} \oplus \cdots \oplus \mathcal{O}_k/\mathfrak{m}^{k_d}$$

for some positive integers $k_1, \dots, k_d$. Hence we see that g is a power of q; then, gh is the number of balls of radius s in $\overline{B}(0,1)^n$. But then $gh = q^{\mu \cdot n}$, so h is a power of q.

If U is any ball and L is a linear isomorphism, then we may reduce to the above case by translating and scaling. Then we have shown that there exists some $r > 0$ such that for all $0 < s \leq r$:

1. $L(U)$ is the disjoint union of balls of radius s.
2. The number of balls in the decomposition is a power of q.

Now let X, $\{U_i\}_{1\leq i\leq n}$ and L_i be as in the statement of the theorem. For $1 \leq i \leq n$, let r_i be the radii satisfying the above conditions for $L_i(U_i)$ and similarly let r' the radius corresponding to X. Let $r = \min\{r', r_1, \ldots, r_n\}$. Then X decomposes into q^m balls of radius r, for some m, and for each $1 \leq i \leq n$, $L_i(U_i)$ decomposes into q^{m_i} balls of radius r, for some m_i. We can then compute:

$$1 \equiv q^m = \sum_{i=1}^{n} q^{m_i} \equiv \sum_{i=1}^{n} 1 = n \mod (q-1)$$

□

The proof of the main theorem is now a series of reductions to this special case. These reductions were only only suggested in the original proof-sketch in [Serre, 1992], but are detailed fully below.

Theorem 3 (Classification of compact manifolds over k)**.** *[Serre, 1992, pp 99 - 100] Let q be the size of the finite residue field of k and X an n-dimensional ($n \geq 1$), compact, non-empty analytic manifold over k. Then:*

1. *X decomposes as a disjoint union of a finite number of balls.*
2. *If $\{U_i\}_{i\in I}$ and $\{V_j\}_{j\in J}$ are two such decompositions of X, then $|I| \equiv |J| \mod q-1$.*

Proof. The first statement follows from theorem 1 and the compactness of X.

For the second statement, we will reduce the situation in the statement of the theorem to the case presented in theorem 2. A key point is that by replacing a ball with q^i balls for any $i \geq 0$, the number of balls modulo $q-1$ is unchanged.

Let $\{U_i\}_{i\in I}$ and $\{V_j\}_{j\in J}$ be two decompositions of the manifold X into disjoint balls. Firstly, we reduce to the case where $\{U_i\}_{i\in I}$ is a refinement of $\{V_j\}_{j\in J}$. Fix a ball U_i so that there exists an analytic isomorphism ϕ_i such that $\phi_i(U_i) \subseteq k^n$ is a ball in k^n. Then $\{V_j\}_{j\in J}$ is an open cover of X, so by letting $W_j = \phi_i(U_i \cap V_j)$, $\{W_j\}_{j\in J}$ forms an open cover of $\phi_i(U_i)$. By Lebesgue's number lemma, there exists some $r > 0$ so that every ball of radius less than or equal to r is contained in W_j for some j. We can then invoke lemma 6 repeatedly to subdivide U_i into q^i balls, for some i, each of which is contained within V_j for some j.

Assuming the theorem holds for $X = V_j$ with decomposition $\{U_a\}_{a\in A}$, and $X' = V_k$ with decomposition $\{U_b\}_{b\in B}$, then the theorem holds for $X \sqcup X'$ since $|\{X, X'\}| = 2 = 1+1 \equiv |A|+|B| \equiv |A \cup B| \mod q-1$. Hence we can assume that $|J| = 1$. By taking charts $(X, \Psi), \{(U_i, \Phi_i)\}_{i\in I}$ we can replace X and each U_i by a ball in k^n. Then there exist analytic isomorphisms $\phi_i : U_i \to \phi_i U_i \subseteq X$, for each $i \in I$, such that X is a disjoint union of the $\phi_i(U_i)$, by taking $\phi_i = \Psi \circ \Phi_i^{-1}$.

Since ϕ_i is analytic, at each $x \in U_i$ there exists a ball B_x with radius r_x such that $\phi_i|_{B_x}$ is given by power series convergent on U_i. Then there exist a finite number of such balls covering U_i by compactness, so by subdividing U_i using lemma 6 we can then assume that ϕ_i is given by a power series on U_i for each $i \in I$.

Note that since ϕ_i is analytic and an isomorphism, the Jacobian is non-vanishing at all points of U and it follows by discreteness of the valuation group that the function $a \mapsto |\det J_{\phi_i}(a)|$ is locally constant. By repeating the previous argument for a suitable choice of balls B_x, we can further assume that for all $x \in U_i$, $|\det J_{\phi_i}(x)|$ is constant.

Fix some $i \in I$. For notational convenience, denote $U = U_i$ and $\phi = \phi_i$. By first translating and scaling U, we can assume that $U = \overline{B}(0,1)^n$.

Let $\phi' = J_\phi(0)^{-1} \circ \phi$. Then ϕ' converges on U but $|\det J_{\phi'}(0)| = 1$. Define a constant

$$m = \max_{\substack{|\alpha| \geq 1 \\ 1 \leq i \leq n}} |c_{i,\alpha}|$$

where $c_{i,\alpha}$ is the coefficient of X^α in the component function ϕ'_i expanded around 0. This is well-defined since $|c_{i,\alpha}| \to 0$ as $|\alpha| \to \infty$ as ϕ'_i converges on $(1,\dots,1)$. We also define r to be the smallest positive integer such that $|\pi^{-r+1}| \geq |m|$.

We now observe that U can be decomposed into q^j translates of the closed ball of radius $|\pi^r|$ around 0, for some j. Let these balls be denoted by $\{B_a\}_{a \in C}$ where $C \subseteq \overline{B}(0,1)^n$ is the set of centers of each ball, so that $|C| = q^j$. Then, replace U by these balls and ϕ by its restriction $\phi|_{B_a}$ to each ball. The claim is now that for each $a \in C$, $\phi|_{B_a}$ can be written as $\phi|_{B_a} = L_a \circ \psi_a$ where L_a is a linear isomorphism and ψ_a is an analytic isomorphism of balls. If this is the case, then we can clearly assume that $\phi_a = L_a$ and conclude using theorem 2.

To prove the claim, fix some $a \in C$. For $b \in k$, we will denote by T_b the translation mapping $x \mapsto x + b$. B_0 denotes the closed ball of radius $|\pi^r|$ around 0 and ψ denotes the map $\phi' \circ T_a$. Then we can write:

$$\begin{aligned} \phi\big|_{B_a} &= J_\phi(0) \circ \psi\big|_{B_0} \\ &= J_\phi(0) \circ T_{\psi(0)} \circ \psi'\big|_{B_0} \end{aligned}$$

where $\psi' = T^{-1}_{\psi(0)} \circ \psi$, so that $\psi'(0) = 0$. Now, for any $\nu \in \mathbb{Z}$, let S_ν denote the linear scaling $x \mapsto \pi^\nu x$, and define $\psi'' = S_r^{-1} \circ \psi' \circ S_r$, where r is as defined above. Then:

$$\phi\big|_{B_a} = J_\phi(0) \circ T_{\psi(0)} \circ S_r \circ \psi'' \circ S_r^{-1}\big|_{B_0}$$

and so it suffices to show that ψ'' is an analytic isomorphism of the ball $S_r^{-1}(B_0) = \overline{B}(0,1)^n$ onto another ball, since then $\phi|_{B_a}$ is written in the desired form.

For each $1 \leq i \leq n$, the power series expansion of the component function ψ'_i is given by:

$$\begin{aligned} \psi'_i(X) &= \sum_{|\alpha| \geq 0} c_{i,\alpha}(X+a)^\alpha \\ &= \sum_{|\alpha| \geq 0} c_{i,\alpha} \prod_{j=1}^{n} \sum_{\beta_j=0}^{\alpha_j} \binom{\alpha_j}{\beta_j} a_j^{\alpha_j - \beta_j} X_j^{\beta_j} \\ &= \sum_{|\beta| > 0} \left(\sum_{|\alpha| \geq |\beta|} \binom{\alpha}{\beta} c_{i,\alpha} a^\alpha \right) X^\beta \end{aligned}$$

where

$$\binom{\alpha}{\beta} = \prod_{j=1}^{n} \binom{\alpha_j}{\beta_j}$$

Note that it is valid to perform this rearrangement by lemma 4: at any $\lambda \in \overline{B}(0,1)^n$, the terms of the series evaluated at λ on the second line are bounded by $|c_{i,\alpha}|$, which tends to 0 independently of β. Denote the coefficients of ψ_i' by $c'_{i,\beta}$. Then we see that, for each i, β:

$$\begin{aligned} |c'_{i,\beta}| &\le \max_{|\alpha|\ge|\beta|} \left| \binom{\alpha}{\beta} c_{i,\alpha} a^{\alpha} \right| \\ &\le \max_{|\alpha|\ge|\beta|} |c_{i,\alpha} a^{\alpha}| \\ &\le \max_{|\alpha|\ge|\beta|} |c_{i,\alpha}| \\ &\le m \end{aligned}$$

since $|a^{\alpha}| \le |(1,\dots,1)^{\alpha}|$. Furthermore, for each i, β with $|\beta| > 0$:

$$|c'_{i,\beta} \cdot \pi^{r(|\beta|-1)}| \le |m| \cdot |\pi^{r(|\beta|-1)}| \le |\pi^{|\beta|-1}|^r \le |\pi^{|\beta|-1}|$$

since $r \ge 1$ and $|\pi^{|\beta|-1}| \le 1$.

So finally, ψ'' has the following properties:

1. $\psi''(0) = 0$
2. $|\det J_{\psi''}(0)| = |\det J_{\phi'}(a)| = 1$
3. Each component function ψ_i'' is a special restricted power series, since its coefficients in the expansion around 0 are given precisely by $c'_{i,\beta} \cdot \pi^{r(|\beta|-1)}$.

Hence, ψ'' satisfies the conditions for lemma 5 so that it defines an analytic isomorphism of $\overline{B}(0,1)^n$ to itself, proving the claim.

$\square$

5 Conclusion

In this note, we have given, by directly working with analytic morphisms over k^n, a proof of Serre's theorem for classifying the structure of compact analytic manifolds over k. A much cleaner proof of this result is possible using the theory of differential forms - see, for example, theorem 7.5.1 in [Igusa, 2000]. However, the approach seen here can be considered to be more explicitly linked to the geometry of k.

This result indicates that a naïve definition of an analytic space over such fields which mirrors the usual definition over $\mathbb{R}$ or $\mathbb{C}$ fails to give us interesting and useful notions of analytic structures such as curves. This provides motivation for the development of new classes of spaces, including Tate's rigid analytic spaces and Berkovich analytic spaces.

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